Representations of the Lie algebra $\widehat{\mathfrak{gl}}_\infty$ and "reciprocity formula" for Clebsch – Gordan coefficients

by BORIS SHOIKHET, e-mail: burman@ium.ips.ras.ru

Introduction

In this article I compare two approaches to the problem of calculating the characters of the Lie algebra $\widehat{\mathfrak{gl}}_\infty$ irreducible representations with "semidominant" highest weights and integral central charge. The first approach is the remarkable result of V. Kac and A. Radul [1], the second one is a small generalization of the author's approach in [2]. As central charge $c \in \mathbb{Z}_{\leq 0}$ tends to $-\infty$, the second approach also gives a precise answer. In this case, both answers represent $\widehat{\mathfrak{gl}}_\infty$-module as a direct sum of the irreducible $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-modules ($\mathfrak{gl}^{(i)}_{\frac{\infty}{2}}$ are natural subalgebras in $\widehat{\mathfrak{gl}}_\infty$), but in completely different form. The equivalence of both formulas gives some "reciprocity formula" for Clebsch – Gordan coefficients (see (6) of §2). These coefficients are defined by the identity $L(\alpha) \otimes L(\beta) = \bigoplus\limits_\gamma C^\gamma_{\alpha\beta} L(\gamma)$, $C^\gamma_{\alpha\beta} \in \mathbb{Z}_{\geq 0}$, for tensor product of two irreducible $\mathfrak{gl}_N$(or $\mathfrak{gl}_{\frac{\infty}{2}}$)-modules with dominant highest weights. (These coefficients depend of N, but they stabilize when N tends to ∞).

I am grateful to S. Arkhipov for useful discussions.

§1 Decomposition of the induced representations

1.1

There are two natural subalgebras, $\mathfrak{gl}^{(1)}_n$ and $\mathfrak{gl}^{(2)}_n$, in the Lie algebra $\mathfrak{gl}_{2n}$ of complex $2n \times 2n$-matrices, and there are also two Abelian subalgebras, $\mathfrak{a}_+$

and $\mathfrak{a}_-$ (see Fig. 1).

$$\begin{array}{|c|c|}\hline \mathfrak{gl}_n^{(1)} & \mathfrak{a}_+ \\ \hline \mathfrak{a}_- & \mathfrak{gl}_n^{(2)} \\ \hline \end{array} \quad 2n$$

Figure 1:

Adjoint action of the subalgebra $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$ preserves the universal enveloping algebra $U(\mathfrak{a}_-) \cong S^*(\mathfrak{a}_-)$, and we want to decompose $S^*(\mathfrak{a}_-)$ with respect to this action. Let $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{gl}_{2n}$ and $\mathfrak{n}_-^{(i)} \oplus \mathfrak{h}^{(i)} \oplus \mathfrak{n}_+^{(i)} = \mathfrak{gl}_n^{(i)}$ be the standart Cartan decompositions, and let E_{ij} be the element of $\mathfrak{gl}_{2n}$ with 1 in the (i,j)-cell and 0 in other cells. Next, denote

$$\mathrm{Det}_k = \mathrm{Det}\begin{pmatrix} E_{n+1,n-k+1} & \dots & E_{n+1,n} \\ \vdots & & \vdots \\ E_{n+k,n-k+1} & \dots & E_{n+k,n} \end{pmatrix} \in U(\mathfrak{a}_-) \cong S^*(\mathfrak{a}_-) \quad (k = 1 \dots n)$$

1.1.1

Lemma. *Monomials* $\mathrm{Det}_1^{l_1} \cdot \ldots \cdot \mathrm{Det}_n^{l_n}$ $(l_i \in \mathbb{Z}_{\geq 0})$ *exhaust all the* $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$*-singular vectors with respect to the adjoint action.*

1.1.2

Lemma. *With respect to the adjoin action of the Lie algebra* $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$, $S^*(\mathfrak{a}_-)$ *decomposes into direct sum of finite-dimensional irreducible modules* $L_w \otimes L_w$ *where* w *goes through all the monomials* $\mathrm{Det}_1^{l_1} \cdot \ldots \cdot \mathrm{Det}_n^{l_n}$, $l_i \in \mathbb{Z}_{\geq 0}$.

We will prove these Lemmas in Section 1.2.

1.1.3

Remark. In the sequel we will need an expression of the highest weight $\theta_{l_1,\dots,l_n}$ of monomial $\mathrm{Det}_1^{l_1} \cdot \ldots \cdot \mathrm{Det}_n^{l_n}$ through the basis $\{E_{ii}\}$ of the Cartan subalgebras

$\mathfrak{h}^{(1)} \oplus \mathfrak{h}^{(2)}$. We have:

$$\begin{aligned}
&\theta_{l_1,\dots,l_n}(E_{11}) = -l_n \\
&\theta_{l_1,\dots,l_n}(E_{22}) = -l_n - l_{n-1} \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\theta_{l_1,\dots,l_n}(E_{nn}) = -l_n - l_{n-1} - \dots - l_1
\end{aligned}$$

and

$$\begin{aligned}
&\theta_{l_1,\dots,l_n}(E_{n+1,n+1}) = l_1 + \dots + l_n \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\theta_{l_1,\dots,l_n}(E_{2n,2n}) = l_n
\end{aligned}$$

The corresponding Young diagram is shown in the Fig. 2:

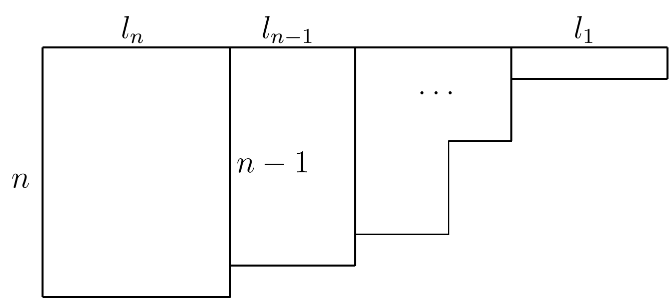

Figure 2:

1.2

In this Section we prove Lemmas 1.1.1 and 1.1.2.

1.2.1

Proof of Lemma 1.1.1: The fact that the monomials $\mathrm{Det}_1^{l_1} \cdot \ldots \cdot \mathrm{Det}_n^{l_n}$ are $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$-singular vectors, is straightforward. Conversely, let ξ be a $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$-singular vector, $\xi \in S^*(\mathfrak{a}_-)$. Consider the minimal rectangular domain in $\mathfrak{a}_-$ with a vertex in the central (upper right) corner which includes all the elements contained in the notation of ξ. Suppose, for example, that its

horizontal side is not smaller then the vertical one, and has the length l. Then there are no more than l squares in the lth column, and elements of $\mathfrak{n}_+^{(i)}$ shift the lth column to 1st, ..., $(l-1)$th. Applying these elements, we should obtain 0, and we obtain $(l-1)$ equations on ξ. Therefore, if ξ is *linear* in elements of the lth column, then $\xi = C \cdot \mathrm{Det}_l$, where C is an expression in a smaller square. Next, all (not only linear) expressions in elements of lth column that vanish under the corresponding $(l-1)$ vector fields, are divisible on some degree of Det_l, say Det_l^k, via arguments of grading, and we have $\xi = C \cdot \mathrm{Det}_l^k$, where C is an expression in a smaller square. We can apply the preceding arguments to the expression C. (see also [2], Ch. I, §1, Sec. 1.3) □

1.2.2

Proof of Lemma 1.1.2. The $\mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)}$-action on $S^*(\mathfrak{a}_-)$ is locally nilpotent. □

1.2.3

In the case $n = \infty$ we obtain the following result:

$$S^*(\mathfrak{a}_-) = \bigoplus_{\substack{\text{for all} \\ \text{Young diagrams } D}} L_D \otimes L_D \tag{1}$$

where L_D is the irreducible $\mathfrak{gl}_{\frac{\infty}{2}}$-module with highest weight θ_D.

1.3

Definition. The weight $\chi\colon \mathfrak{h} \to \mathbb{C}$ is a *semidominant*, if its restrictions $\chi^{(1)}$ and $\chi^{(2)}$ on subalgebras $\mathfrak{gl}_n^{(1)}$ and $\mathfrak{gl}_n^{(2)}$ are dominant weights. In other words, in the basis $\{E_{ii}\}$ of $\mathfrak{h}$ we have $\chi = (\chi_1, \dots, \chi_n)$ where all the χ_i belong to $\mathbb{Z}$, $\chi_1 \geq \chi_2 \geq \dots \geq \chi_n$, and $\chi_{n+1} \geq \dots \geq \chi_{2n}$.

Difference $\chi_n - \chi_{n+1}$ may be a negative integer.

1.3.1

Let $\chi\colon \mathfrak{h} \to \mathbb{C}$ be a semidominant weight, and let $L_{\chi^{(1)}} \otimes L_{\chi^{(2)}}$ be corresponding (finite dimensional) irreducible $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$-module. We can continue this

module on subalgebra $\gamma = \mathfrak{h} + \mathfrak{a}_+ \oplus \mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$ by formulas $hv = \chi(h)v$ for any $h \in \mathfrak{h}$, and $\mathfrak{a}_+ v = 0$.

Definition. $\mathrm{Ind}_\chi = U(\mathfrak{gl}_{2n}) \bigotimes\limits_{U(\gamma)} (L_{\chi^{(1)}} \otimes L_{\chi^{(2)}})$.

1.3.2

Lemma. *As a* $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$*-module,* Ind_χ *is* $(L_{\chi^{(1)}} \otimes L_{\chi^{(2)}}) \bigotimes S^*(\mathfrak{a}_-)$.

In particular, Ind_χ decomposes into the direct sum of finite-dimensional irreducible $\mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)}$-modules.

Proof: It is obvious. □

1.3.3

Consider the case of $\widehat{\mathfrak{gl}}_\infty$.

With all the values c of central charge and χ_{centr} of highest weight χ on "central" coroot, as $\mathfrak{gl}_{\frac{\infty}{2}}^{(1)} \oplus \mathfrak{gl}_{\frac{\infty}{2}}^{(2)}$-module.

$$\mathrm{Ind}_{\chi,c} = \bigoplus_{\substack{\text{for all} \\ \text{Young diagrams } D}} (L_{\chi^{(1)}} \otimes L_{\chi^{(2)}}) \bigotimes (L_D \otimes L_D) =$$

$$= \bigoplus_D (L_{\chi^{(1)}} \otimes L_D) \bigotimes (L_{\chi^{(2)}} \otimes L_D) \quad (2)$$

We have:

$$L_\alpha \otimes L_\beta = \mathop{\oplus}_{\gamma} C_{\alpha\beta}^{\gamma} L_\gamma, \quad (3)$$

where L_α, L_β, L_γ are irreducible $\mathfrak{gl}_{\frac{\infty}{2}}$-modules with highest weights α, β, γ; $C_{\alpha\beta}^{\gamma} \in \mathbb{Z}_{\geq 0}$ are called Clebsch – Gordan coefficients.

Therefore, as $\mathfrak{gl}_{\frac{\infty}{2}}^{(1)} \oplus \mathfrak{gl}_{\frac{\infty}{2}}^{(2)}$-module,

$$\mathrm{Ind}_{\chi,c} = \bigoplus_{D,\nu_1,\nu_2} C_{\chi^{(1)},D}^{\nu_1} C_{\chi^{(2)},D}^{\nu_2} L_{\nu_1} \otimes L_{\nu_2} \quad (4)$$

In particular, the multiplicity of $\mathfrak{gl}_{\frac{\infty}{2}}^{(1)} \oplus \mathfrak{gl}_{\frac{\infty}{2}}^{(2)}$-module $L_{\nu_1} \otimes L_{\nu_2}$ in $\mathrm{Ind}_{\chi,c}$ is equal to

$$\sum_{\text{for all } D} C_{\chi^{(1)},D}^{\nu_1} \cdot C_{\chi^{(2)},D}^{\nu_2} \quad (5)$$

Note, that the sum is finite.

§2 Irreducible representations of the Lie algebra $\widehat{\mathfrak{gl}}_\infty$ with $c = -N$ when $N \to \infty$

2.1

In §1 we decomposed the $\widehat{\mathfrak{gl}}_\infty$-module $\mathrm{Ind}_{\chi,-N}$ into the direct sum of irreducible $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-modules, and this decomposition does not depend on central charge c and $\chi_{centr} = \chi(\alpha_0^\vee)$; and it is interesting to find irreducible factor of $\mathrm{Ind}_{\chi,-N}$ in the same terms, i.e. "point out" $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-modules, which are present in this irreducible factor, with dependence on c and χ_{centr}. This is very interesting, and, I think, very difficult problem. It was solved in the author's paper [2] only in a particular case, when $\chi = 0$ and any $c \in \mathbb{C}$ (see 2.1.5 and [2], Ch. I, §2).

However, when $c = -N \in \mathbb{Z}_{\leq 0}$ and $N \to \infty$, corresponding irreducible factor of the module $\mathrm{Ind}_{\chi,-N}$ tends to the whole module $\mathrm{Ind}_{\chi,-N}$. Comparing this fact with the result of V. Kac and A. Radul [1] we obtain relations on Clebsch – Gordan coefficients.

2.1.1

Theorem. *Let central charge $c = -N$ ($N \gg 0$). Then there exists $\delta(N) \in \mathbb{Z}_{\geq 0}$ which tends to ∞ as N tends to ∞, and such that there are no $\widehat{\mathfrak{gl}}_\infty$-singular vectors on all the levels of the representation $\mathrm{Ind}_{\chi,-N}$, which are less then $\delta(N)$.*

Corollary. *Maximal submodule in $\mathrm{Ind}_{\chi,-N}$ does not intersect with levels less than $\delta(N)$.* □

Only $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-singular vector may be $\widehat{\mathfrak{gl}}_\infty$-singular vector, this vector is equal to the sum of the $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-singular vectors in several representations $(L_{\chi^{(1)}} \otimes L_D) \bigotimes (L_{\chi^{(2)}} \otimes L_D)$. We want to show, that diagram D should be very big when $N \gg 0$.

2.1.2

Lemma. *Let V, W be two irreducible $\mathfrak{gl}_N$(or $\mathfrak{gl}_{\frac{\infty}{2}}$)-modules, let v and w be their highest vectors. Then every singular vector $\sum_i \theta_1^{(i)} v \otimes \theta_2^{(i)} v$ contains the term with $\theta_1 = 1$.*

Proof: V and W are irreducible representations, and hence they do not contain singular vectors besides their highest weight vectors. $\square$

2.1.3

Let $e_0 \in \mathfrak{n}_+ \subset \mathfrak{gl}_{2n}(\widehat{\mathfrak{gl}}_\infty)$ be the "corner" element of $\mathfrak{a}_+$ (see Fig. 1), in the case of $\mathfrak{gl}_{2n}$, $e_0 = E_{n,n+1}$; and let $\alpha_0^\vee$ be "central" coroot, $\chi(\alpha_0^\vee) = \chi_{centr}$. We will need in the sequel the direct expression for the commutator $[e_0, \mathrm{Det}_k^l]$. For this, introduce the following notations. Denote by $\{y_{ij}\}$ the elements of $\mathfrak{a}_-$, in the case of $\mathfrak{gl}_{2n}$ $y_{ij} = E_{n+i,n-j+1}$; we will use these notations also in the case of $\widehat{\mathfrak{gl}}_\infty$. Denote $A_k = (y_{ij})_{i,j=1\dots k}$, $\widetilde{A}_k = (y_{ij})_{i,j=2\dots k}$ (we have $\mathrm{Det}_k = \mathrm{Det} A_k$). Let $\widetilde{A}_{ij}$ be the matrix $\widetilde{A}_k$ without jth row and ith column. Let z_i^+ be the element from $\mathfrak{n}_-^{(1)}$, standing in the ith column above y_{1i}, and z_j^- be an element from $\mathfrak{n}_-^{(2)}$, standing in the jth row right from y_{j1}. Now we are ready to formulate the result:

Lemma.

$$[e_0, \mathrm{Det}_k^l] = (-1)^{1+k} \cdot l \cdot (\mathrm{Det}\widetilde{A}_k \cdot \mathrm{Det}_k^{l-1}(\alpha_0^\vee + c + k - l) + \\ + \sum_{i,j=2\dots k} (-1)^{i+j} \cdot l \cdot \mathrm{Det}\widetilde{A}_{ij} \cdot \mathrm{Det}_k^{l-1}(y_{j1} z_i^+ - y_{1i} z_j^-)$$

Proof. It is a direct calculation from [2], Ch. I, §1. $\square$

Note, that the sum belongs to $U(\mathfrak{a}_-)\mathfrak{n}_-^{(1)} \oplus U(\mathfrak{a}_-)\mathfrak{n}_-^{(2)}$.

2.1.4

The proof of the Theorem: Any $\widehat{\mathfrak{gl}}_\infty$-singular vector is represented as

$$\theta = \sum_D \sum_i [n_{s_i}^{i,D}[\cdots [n_1^{i,D}, D] \cdots] \cdot n_{s_{i+1}}^{i,D} \cdot \ldots \cdot n_{r_i}^{i,D} \cdot v,$$

where $n_j^{i,D} \in \mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)}$ and v is the highest weight vector. We want to prove that $[e_0, \theta] \neq 0$ for $N \gg 0$.

(1) e_0 commutes with $\mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)}$, and so,

$$[e_0, \theta] = \sum_D \sum_i [n_{s_i}^{i,D}[\cdots [n_1^{i,D}[e_o, D] \cdots] n_{s_{i+1}}^{i,D} \cdot \ldots \cdot n_{r_i}^{i,D} \cdot v$$

(2) by Lemma 2.1.2, the notation of θ contains the term $D \cdot \alpha v$, where $\alpha \in U(\mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)})$, we want to prove, that the term $[e_0, D]\alpha \cdot v$ can not annihilate with other terms

(3) We will consider only the case $D = \mathrm{Det}_k^l$, the general case is similar

(4) Denote $\widetilde{D} = \mathrm{Det}\widetilde{A}_k \mathrm{Det}_k^{l-1}$; by Lemma 2.1.3,

$$[e_0, \mathrm{Det}_k^l] = (-1)^{1+k} \cdot l \cdot \widetilde{D}(\alpha_0^\vee + c + k - l) + o(N),$$

the first term is very big, because c acts as multiplication by $-N$

(5) For $n \in \mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)}$ we have:

$$[n, \widetilde{D}(\alpha_0^\vee + c + k - l)] = [n, \widetilde{D}](\alpha_0^\vee + c + k - l) + \widetilde{D}[n, \alpha_0^\vee];$$

the second summand is $o(N)$, and $[n, \widetilde{D}]$ is not equal to $\widetilde{D}$ element of $U(\mathfrak{a}_-)$

□

2.1.5

Remark. It is possible to get the explicit answer for the $\widehat{\mathfrak{gl}}_\infty$-module Ind_μ (with $\chi = 0$ and $c = \mu \in \mathbb{C}$): when $\mu = N \in \mathbb{Z}_{\geq 0}$, the singular vectors are $\mathrm{Det}_1^{N+1} v$, $\mathrm{Det}_2^{N+2} v$, $\ldots$, and when $\mu = -N \in \mathbb{Z}_{\leq 0}$, the singular vectors are $\mathrm{Det}_{N+1} v$, $\mathrm{Det}_{N+2}^2 v$, $\ldots$. We see that irreducible factor of Ind_μ ($\mu = \pm N$) tends to Ind_μ when $N \to \infty$. When $\mu \notin \mathbb{Z}$ there are no singular vectors in Ind_μ, and Ind_μ is irreducible. This is a result from [2], Ch. I, §1.

2.2

In this Section, we compare Theorem 2.1.1 with result from the paper by V. Kac and A. Radul [1].

2.2.1

We define a set

$$H_N = \{(\nu_1, \nu_2, \ldots, \nu_N),\ \nu_1 \geq \nu_2 \geq \ldots \geq \nu_N,\ \nu_i \in \mathbb{Z} \quad \text{for all} \quad i = 1 \ldots N\},$$

H_N^+ (H_N^-) is a subset in H_n which consists of $\nu_i \in \mathbb{Z}_{\geq 0}$ ($\nu_i \in \mathbb{Z}_{\leq 0}$). Let $\nu \in H_N$; define $\widehat{\mathfrak{gl}}_\infty$-weight $\Lambda(\nu)$ as follows:

$$\Lambda(\nu) = (\ldots 0, 0, \nu_{p+1}, \ldots, \nu_N; \nu_1, \nu_2, \ldots, \nu_p, 0, 0, \ldots),$$

where $\nu_1, \ldots, \nu_p \geq 0$, $\nu_{p+1}, \ldots, \nu_N \leq 0$, and we put a semicolon between the 0th and the first slot. So, $\Lambda_-(\nu) = (\ldots 0, 0, \nu_{p+1}, \ldots, \nu_N)$ is the $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}}$-weight and $\Lambda_+(\nu) = (\nu_1, \nu_2, \ldots, \nu_p, 0, 0, \ldots)$ is the $\mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$-weight.

Theorem ([1]). *As* $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}} \oplus \mathfrak{gl}^{(2)}_{\frac{\infty}{2}}$*-module,*

$$L(\Lambda(\nu), -N) = \bigoplus_{\substack{\lambda \in H_N^- \\ \mu \in H_N^+}} C^{\nu}_{\lambda,\mu} L_-(\lambda) \otimes L_+(\mu).$$

Comments. $L(\Lambda(\nu), -N)$ is the irreducible $\widehat{\mathfrak{gl}}_\infty$-module with the highest weight $\Lambda(\nu)$ and the central charge $-N$; where $C^{\nu}_{\lambda\mu}$ are Clebsch – Gordan coefficients for the Lie algebra $\mathfrak{gl}_N$ (see (3)); $L_-(\lambda)(L_+(\mu))$ are the irreducible $\mathfrak{gl}^{(1)}_{\frac{\infty}{2}}(\mathfrak{gl}^{(2)}_{\frac{\infty}{2}})$-modules with highest weights λ (resp. μ).

2.2.2

Send N to ∞, and choose a $\mathfrak{gl}_N$-weight ν such that

$$\nu = (\nu_1, \ldots, \nu_k, 0, 0 \ldots\ldots, 0, 0, \nu_s, \ldots, \nu_N)$$

where $\nu_1, \ldots, \nu_k > 0$, $\nu_s, \ldots, \nu_N < 0$; as N grows we just add zeros between them.

Theorem.

$$\sum_{\substack{\text{for all} \\ \text{Young diagrams } D}} C^{\lambda_-}_{\Lambda_-(\nu), D_-} \cdot C^{\mu_+}_{\Lambda_+(\nu), D_+} \quad (\textit{in the sense of } \mathfrak{gl}_{\frac{\infty}{2}}) =$$

$$= C^{\nu}_{\lambda_-, \mu_+} \quad (\textit{in the sense of } \mathfrak{gl}_N,\ N \gg 0,\ \lambda_-,\ \mu_+, \textit{ and } \nu \textit{ are fixed}) \quad (6)$$

Comments. If D is as in Fig. 1, then

$$D_- = \theta_{D_-} = (\ldots, 0, 0, -l_n, \ldots, -l_2 - \ldots - l_n, -l_1 - \ldots - l_n)$$

and

$$D_+ = \theta_{D_+} = (l_1 + \ldots + l_n, l_2 + \ldots + l_n, \ldots, l_n, 0, 0, \ldots);$$

$\Lambda_+(\nu)$ and $\Lambda_-(\nu)$ were defined in 2.2.1. The sum in the left-hand side of (6) is finite. The right-hand side stabilizes, as N grows.

Proof: it is an obvious consequence of Theorems 2.1.1 and 2.2.1 and (5) of §1. $\square$

2.2.3

Remark. For the tautological $\mathfrak{sl}_N$-module V, tensor products $V \otimes V$ and $V \otimes V^*$ are not isomorphic:

$$V \otimes V = S^2(V) \oplus \Lambda^2(V),$$
$$V \otimes V^* = \mathfrak{sl}_N \oplus \mathbb{C}$$

The $\mathfrak{gl}_N$-weights λ_- (resp. μ_+) define representations in tensor powers of V^* (resp. V), and weight ν defines "mixed" representation.

References

[1] V. Kac and A. Radul. *The vertex algebra* $W_{1+\infty}$, Transf. groups, vol. **1**, Numbers 1&2, 1996

[2] B. B. Shoikhet. *Certain topics on a Lie algebra* $\mathfrak{gl}(\lambda)$ *representation theory*, Transl. from: Itogy Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. **38**, Plenum Publ., 1996